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## A NOTE ON THE TOTAL UNIMODULARITY OF MATRICES

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A necessary and sufficient characterization of totally unimodular matrices is given which is derived from a necessary condition for total unimodularity due to Camion. This characterization is then used in connection with a theorem of Hoffman and Kruskal to provide an elementary proof of the characterization of totally unimodular matrices in terms of forbidden submatrices due to Camion.

For any  $m \times n$  matrix  $A$  of integers and any vector  $b$  of  $m$  integers let

$$P(A, b) = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\},$$

$$P_1 = \text{conv}\{P(A, b) \cap \mathbb{Z}^n\},$$

denote the polytopes in real  $n$ -space  $\mathbb{R}^n$  defined by the linear system of inequalities  $Ax \leq b, x \geq 0$ , and the convex hull of integer solutions to  $Ax \leq b, x \geq 0$ , respectively. It is of prime interest in integer programming to know under what conditions on  $A$  and  $b$ , respectively, the two polytopes  $P(A, b)$  and  $P_1(A, b)$  coincide, i.e.,  $P(A, b) = P_1(A, b)$ . If the matrix  $A$  is totally unimodular, i.e. if every minor of  $A$  has a determinant of  $0, \pm 1$ , then it follows by Cramer's rule that  $P(A, b) = P_1(A, b)$  no matter what integer vector  $b$  is used in the definition of  $P(A, b)$ . This is the easy part of the well-known theorem of Hoffman and Kruskal [5], for which in [3], see also [4], a short and elementary proof has been given.

**Theorem 1** (Hoffman and Kruskal [5]). *Let  $A$  be any  $m \times n$  matrix of integers and  $b$  be any vector of  $m$  integers. Then the following two statements are equivalent.*

- (i)  $P(A, b) = P_1(A, b)$  for all  $b \in \mathbb{Z}^m$ .
- (ii)  $A$  is totally unimodular.

The notion of totally unimodular matrices has recently been extended — in the case of zero-one matrices — to the class of *balanced* matrices [1] and *perfect* matrices [6]. See [7] for a survey relating the three concepts and known characterizations of the respective matrices.

Since the total unimodularity of a matrix  $A$  implies and is implied by the total unimodularity of the  $2(m+n) \times n$  matrix whose rows correspond to  $A$ ,  $-A$ ,  $I$  and  $-I$ , respectively, where  $I$  is the  $n \times n$  identity matrix, condition (i) of Theorem 1 can be reformulated equivalently as follows (see [5]): Let

$$Q(A, b^1, b^2, d^1, d^2) = \{x \in \mathbb{R}^n : b^1 \leq x \leq b^2, d^1 \leq x \leq d^2\}$$

and  $Q_1(A, b^1, b^2, d^1, d^2)$  denote its associated convex hull of integer solutions, where  $b^i \in \mathbb{Z}^m$ ,  $d^i \in \mathbb{Z}^n$  are arbitrary integer vectors for  $i = 1, 2$ .

Then statement (i) of Theorem 1 is equivalent to the statement

(i')  $Q(A, b^1, b^2, d^1, d^2) = Q_1(A, b^1, b^2, d^1, d^2)$  for all  $b^i \in \mathbb{Z}^m$  and  $d^i \in \mathbb{Z}^n$ ,  $i = 1, 2$ .

Using the Hoffman—Kruskal theorem we will now prove a characterization of totally unimodular matrices. The result actually is a strengthening of a proposition [2, Statement 3, p. 1071] due to Camion and we will use it to derive Camion's characterization of totally unimodular matrices in terms of forbidden submatrices, thereby establishing a link between the "geometric" approach of Hoffman and Kruskal and the purely algebraic work of Camion.

**Theorem 2.** *Let  $A$  be any  $m \times n$  matrix of integers. Then the following two statements are equivalent.*

- (i)  $A$  is totally unimodular.
- (ii) For all vectors  $x \in \{0, \pm 1\}^n$  there exists a vector  $y \in \{0, \pm 1\}^n$  such that  $y \equiv x \pmod{2}$  and  $A^i y = 0$  if  $A^i x \equiv 0 \pmod{2}$ ,  $A^i y = \pm 1$ , otherwise; where  $A^i$  is the  $i$ th row of  $A$ .

**Proof.** To prove (i)  $\Rightarrow$  (ii) let  $x \in \{0, \pm 1\}^n$  and  $Ax = a$ . Define  $d^v$  for  $v = 1, 2$  to be

$$d_i^v = \begin{cases} 0 & \text{if } x_i \equiv 0 \pmod{2}, \\ \frac{1}{2}(x_i - 1) & \text{if } x_i \equiv 1 \pmod{2} \text{ and } v = 1, \\ \frac{1}{2}(x_i + 1) & \text{if } x_i \equiv 1 \pmod{2} \text{ and } v = 2; \end{cases}$$

and define  $b^v$  for  $v = 1, 2$  to be

$$b_i^v = \begin{cases} \frac{1}{2}a_i & \text{if } a_i \equiv 0 \pmod{2}, \\ \frac{1}{2}(a_i - 1) & \text{if } a_i \equiv 1 \pmod{2} \text{ and } v = 1, \\ \frac{1}{2}(a_i + 1) & \text{if } a_i \equiv 1 \pmod{2} \text{ and } v = 2. \end{cases}$$

The polyhedron  $Q = Q(A, b^1, b^2, d^1, d^2)$  contains  $\frac{1}{2}x$ ; consequently, since  $Q \neq \emptyset$ , it follows from the theorem of Hoffman and Kruskal that there exists a  $x' \in \mathbb{Z}^n \cap Q$ , and hence  $y = x - 2x'$  satisfies condition (ii).

To prove the implication (ii)  $\Rightarrow$  (i) we use induction over  $k$ , the size of  $k \times k$  minors of  $A$ . Choosing  $x = e_j$ , the  $j$ th unit vector in  $\mathbb{Z}^n$ , implies  $y = \pm e_j$  and hence,  $a_{ij} \in \{0, \pm 1\}$  for all coefficients  $a_{ij}$  of  $A$ . Suppose now that all  $k \times k$  minors of  $A$  have determinants of  $0, \pm 1$ . Let  $B$  be any nonsingular minor of  $A$  of size  $(k+1) \times (k+1)$ . Without loss of generality, let the columns of  $B$  coincide with the  $(k+1)$  first columns of  $A$ . By induction hypothesis, all proper minors of  $B$  have determinants of  $0, \pm 1$ . Consequently, letting  $d = \det B$  and  $B^* = dB^{-1}$ , it follows by Cramer's rule that  $B^*$  has only  $0, \pm 1$  entries. Let  $b^1$  be the first column of  $B^*$ . Then,  $Bb^1 = d\tilde{e}_1$ , where  $\tilde{e}_1$  is the first unit vector having  $k+1$  components.

Since  $b^1$  is a vector of  $0, \pm 1$ , let  $x = (b^1, 0)$ , where  $0$  is the  $(n-(k+1))$  zero-vector. Then there exists a  $y \in \{0, \pm 1\}^n$  such that  $y \equiv x \pmod{2}$ . We can write  $y = (y^1, 0)$ , where  $y^1 \neq 0$  is a  $(k+1)$ -vector satisfying condition (ii). Consequently, we have  $By^1 = \pm \tilde{e}_1$ , since  $|B| \neq 0$  and  $y^1 \neq 0$  imply  $By^1 \neq 0$  and hence,  $d \not\equiv 0 \pmod{2}$ . But  $Bb^1 = d\tilde{e}_1$ , and consequently  $d = \pm 1$ .

**Remark 3.** Condition (ii) of Theorem 2 is equivalent to the statement:

(ii') For all  $x \in \mathbb{Z}^n$  there exists a  $y \in \{0, \pm 1\}^n$  such that  $y \equiv x \pmod{2}$  and  $A^i y = 0$  if  $A^i x \equiv 0 \pmod{2}$ ,  $A^i y = \pm 1$ , otherwise; where  $A^i$  is the  $i$ th row of  $A$ .

To prove Remark 3, we note that (i) implies (ii') by the same proof as used to show the implication (i)  $\Rightarrow$  (ii). Since (ii') implies (ii), the equivalence follows.

The algebraic characterization of totally unimodular matrices uses the notion of *eulerian* matrices: A  $k \times k$  matrix  $A$  with coefficients  $a_{ij} \in \{0, \pm 1\}$  is called *eulerian* if and only if  $\sum_{i=1}^k a_{ij} \equiv 0 \pmod{2}$  for  $j = 1, \dots, k$ , and  $\sum_{j=1}^k a_{ij} \equiv 0 \pmod{2}$  for  $i = 1, \dots, k$ .

**Lemma 4** (see [2, Statement 2]). *Let  $A$  be any matrix with coefficients  $a_{ij} \in \{0, \pm 1\}$  and suppose that every eulerian submatrix of  $A$  is singular. Then every nonsingular submatrix  $B$  of  $A$  satisfies  $\det B \equiv 1 \pmod{2}$ .*

**Proof.** Suppose not; then there exists a nonsingular submatrix  $B$  of  $A$  of minimal order such that  $b = \det B$  satisfies  $b \equiv 0 \pmod{2}$ . Since  $B$  is of minimal order, it follows by Cramer's rule that every coefficient of  $bB^{-1}$  is either zero or odd. Let  $x$  be any column of  $bB^{-1}$  and rearrange the components so that  $x^T = (x^1, 0)^T$  has non-zero components in  $x^1$ . Then for the corresponding submatrix  $B^1$  of  $B$  we have  $B^1 x^1 = Bx = be_k$  where  $e_k$  is a unit vector and furthermore,  $x = e + 2w$  where  $w$  is an integer vector.

Since  $|B| \neq 0$ , there exists a nonsingular submatrix  $B''$  of  $B^1$  such that  $B'' x^1 = 2v$  where  $2v$  is a submatrix of  $be_k$  chosen compatibly with  $B''$ . But then  $x^1 = (2/\det B'')h$  with  $h$  an integer vector implies that  $\det B'' \equiv 0 \pmod{2}$ , since  $x^1$  has only odd components. Consequently, since  $B$  was of minimal order, it follows that  $B = B''$  and  $Be \equiv 0 \pmod{2}$ , since  $Bx = Be + 2Bw \equiv 0 \pmod{2}$ . Since we can apply the same reasoning to  $B^T$ , it follows that  $B$  is eulerian and hence by assumption, singular.

**Theorem 5** (see [2, Theorem 1]). *Let  $A$  be any  $m \times n$  matrix with coefficients  $a_{ij} \in \{0, \pm 1\}$ . Then the following two statements are equivalent.*

- (i)  $A$  is totally unimodular.
- (ii) Every eulerian submatrix of  $A$  is singular.

**Proof.** The implication (i)  $\Rightarrow$  (ii) follows from Theorem 1. For suppose that  $A$  has a nonsingular eulerian submatrix  $A'$ . Let  $J$  be the column set of  $A'$ , define  $x$  by setting  $x_j = 1$  for  $j \in J$  and  $x_j = 0$  otherwise, and set  $Ax = 2b^1$ , where  $b^1 \in \mathbb{Z}^m$ . Then the polytope  $Q$  defined above with  $b^1 = b^1$ ,  $d^1 = 0$ ,  $d^2 = e$  is nonempty and has a fractional vertex. Consequently,  $A$  cannot be totally unimodular.

To prove the implication (ii)  $\Rightarrow$  (i), we show that (ii) implies (ii) of Theorem 2. Suppose that this is not the case and let  $x \in \{0, \pm 1\}^n$  be a vector with *minimal support*, i.e. minimal number of non-zeroes, for which Theorem 2, part (ii), is violated. We can assume without loss of generality that the  $k$  first components  $x^1$  of  $x$  are all non-zero, i.e.,  $x = (x^1, 0)$ . Correspondingly, let  $A = (B, R)$  where  $B$  is of size  $m \times k$ . By the minimality assumption every proper submatrix of  $B$  is totally unimodu-

lar, since Theorem 2(ii), is satisfied. By the same argument,  $B$  cannot be totally unimodular. Hence  $B$  contains a  $k \times k$  submatrix  $C$  such that  $c = \det C$  satisfies  $c \notin \{0, \pm 1\}$  and by Lemma 4,  $c \equiv 1 \pmod{2}$ .

Let  $C^* = cC^{-1}$ . Suppose that  $C^*$  contains a zero entry and let  $z$  be a column of  $C^*$  in which a zero occurs. Since the components of  $z$  are all equal to 0 or  $\pm 1$  with at least one component equal to zero, it follows by the same argument that was used in the proof of Theorem 2 from  $Cz = ce_k$ , where  $e_k$  is a unit vector, that  $c = \pm 1$ . Consequently, all entries of  $C^*$  equal  $+1$  or  $-1$  and hence  $C^*e = ke - 2w$ , where  $e$  is the vector of  $k$  ones and  $w$  is an integer vector. Consequently,  $k(Ce) = ce + 2v$ , where  $v$  is an integer vector. Since  $c \equiv 1 \pmod{2}$ , it follows that  $k \equiv 1 \pmod{2}$ . Now let  $D = E - I$ , where  $E$  is the  $k \times k$  matrix of ones and  $I$  is the  $k \times k$  identity matrix. Then it follows from the above that  $C^*D = 2F$ , where  $F$  is the  $k \times k$  matrix of integers. Consequently,  $cD = 2(CF)$  implies  $c \equiv 0 \pmod{2}$ , a contradiction; hence  $B$  cannot contain a submatrix  $C$  with  $\det C \neq 0, \pm 1$ .

Let  $A'$  be any  $k \times k$  eulerian submatrix of a totally unimodular matrix  $A$ . Then by Theorem 2 it follows that there exists a vector  $y \in \{0, \pm 1\}^k$ ,  $y \neq 0$ , such that thus providing an alternative proof of the implication (i)  $\Rightarrow$  (ii) of Theorem 5. Furthermore, writing the vector  $y$  in the form  $y = e - 2y'$ , where  $e$  is the vector having  $k$  components equal to  $+1$ , we obtain  $A'e = 2A'y'$  and consequently,  $e^T A'e \equiv 0 \pmod{4}$ . Hence, the sum of the coefficients of every eulerian submatrix of a totally unimodular matrix is a multiple of 4, which is the easy part of [2, Theorem 2].

It should be noted that Lemma 4 is due to R. Gomory, see [2], who also proved that a  $(0, \pm 1)$ -matrix  $A$  is totally unimodular if and only if  $A$  has no minor whose determinant equals  $\pm 2$  (see [2, Statement 1]). This latter statement requires a proof involving elementary row and column operations on matrices of 0 and  $\pm 1$  and provides in conjunction with Lemma 4 an alternative proof of Theorem 5 which can be found in full in [2]. Gomory's result is of particular interest as neither balanced nor perfect matrices [6] can be characterized by means of forbidden determinantal values; rather, as indicated in [6], given any natural number  $k$ , there exists a perfect matrix having a minor with determinant equal to  $\pm k$ . A similar statement is true for balanced matrices, as one can readily generalize the example given in [1] to verify this statement.

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